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On the creeping motion of two arbitrary-sized touching spheres in a linear shear field

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The Stokes equations describing the creeping motion of two arbitrary-sized touching spheres are solved exactly through the use of tangent-sphere coordinates. For the case of a linear shear field at infinity, explicit results covering the entire range of size ratios are presented for: (*a*) the forces and torques on the aggregate; (*b*) the hydrodynamic forces on the individual spheres comprising a freely suspended aggregate, which are in general non-zero; (*c*) the contribution of the pair to the bulk stress of a dilute suspension; and (*d*) under free suspension conditions, the velocity of any material point relative to that of the undisturbed flow.

1. Introduction

Following Einstein's derivation of his classical expression for the effective viscosity of an infinitely dilute suspension of rigid spheres, considerable effort has been directed towards determining theoretically the rheological properties of two-phase systems consisting of small particles freely suspended in a continuum fluid. Specifically, numerous authors have dealt with the problem originally considered by Einstein and have attempted to extend the analysis to the non-dilute case. In fact, the most common approach along these lines has been to try and determine the coefficient of the $O(c^2)$ term in the Einstein equation

$$\mu^*/\mu = 1+\tfrac{5}{2}c+O(c^2), \tag{1.1}$$

where μ^* is the effective viscosity of the suspension, μ the viscosity of the pure fluid and c the volume fraction of the solid spheres, because, according to experimental evidence, knowledge of this coefficient, whose numerical value is believed to be of $O(10)$, would extend the applicability of (1.1) to values of c as large as 0.2. Otherwise, (1.1) remains of limited usefulness since it is known to become inaccurate when c exceeds approximately 0·03.

Although it has been established for some time that the $O(c^2)$ term in (1.1) reflects the contribution of all first-order particle interactions to the rate of energy dissipation in the suspension, a rigorous theory for computing its coefficient was not developed until recently (Batchelor & Green 1972*b*). An important aspect of this theory, which has already yielded explicit results for the case of equal-sized spheres (Batchelor & Green 1972*b*), is that it can be applied to suspensions containing spheres of different sizes and thereby yield the first theoretical expression for the effective viscosity of such systems beyond the range of infinite dilution, provided that, *inter alia*, detailed information regarding the motion of a pair of spheres freely suspended in a linear shear field under creeping-flow conditions is already available. Thus, problems involving such motions of two interacting spheres are worth studying, not only because of their fundamental nature, but also because, in view of Batchelor & Green's recent theory, they constitute one of the key intermediate steps for the derivation of a significant result in the field of suspension rheology, namely the extension of (1.1) to $O(c^2)$.

Such types of two-sphere problems are of course not new in that they have attracted the attention of numerous investigators for over half a century. Many of these contributions were recently reviewed by Lin, Lee & Sather (1970), who, through the use of bispherical co-ordinates, introduced by Jeffery (1922), also obtained an exact solution to the Stokes equations for the motion of two unequal-sized spheres arbitrarily oriented with respect to a shear field. This solution, given in detail for spheres of equal size, is of particular value to the study of suspension rheology in that, by applying the condition of zero net force and torque to each of the two spheres, it can yield in principle their translational and rotational velocities and thereby their trajectories of free motion throughout the flow field.

In a recent study, Batchelor & Green (1972*a*) made use of this solution by Lin *et al.* (1970) to compute the relative trajectories of two equal-sized spheres freely suspended in a simple shear flow and found, in agreement with earlier experimental observations by Darabaner & Mason (1967), that, depending on their initial location, the two spheres either pass by one another or orbit about each other indefinitely. Batchelor & Green also showed that the shapes of these open and closed trajectories resemble the streamline pattern around a single sphere freely suspended in a simple shear field (Cox, Zia & Mason 1968), and that the kinematics of a pair of spheres, in either orbit, are governed instantaneously by Jeffery's (1922) equations of motion for a spheroid having an axis ratio r_e whose value, however, is constantly changing with the position of the spheres.

The solution given by Lin *et al.* (1970), though applicable in principle for distances between the centres of the spheres of arbitrary magnitude, requires the evaluation of a complicated infinite series involving bispherical harmonics, the convergence of which becomes slower as the minimum clearance between the spheres is reduced. In fact, the solution ceases to apply in the limit of zero clearance since the bispherical co-ordinate system is inherently unable to describe finite-sized spheres in contact. Consequently, although it should be possible to describe qualitatively the motion of touching spheres and to calculate integral quantities (such as forces and torques) by asymptotic considerations, it would be of interest to obtain the exact solution for touching spheres since, as shown recently by Batchelor & Green (1972*a*), such a solution can be used to provide information about some of the functions describing a two-sphere encounter in shear flow when the spheres are close together. Moreover, the exact determination of the flow field around arbitrary-sized touching spheres would be important for evaluating the contribution of such doublets, as well as of certain non-touching pairs in close proximity, to the stress–strain relation that applies for suspensions of solid spheres, and, as mentioned earlier, should prove useful in calculating the coefficient of the $O(c^2)$ term in (1.1) for suspensions containing spheres of mixed sizes. Besides, the present analysis apparently produces the first creeping-flow solution involving a body of revolution without fore-and-aft symmetry, and as such should prove to be of some special interest.

It is the purpose then of this paper to develop a solution for the quasi-steady creeping motion of two arbitrary-sized touching spheres suspended in a linear shear field. This will be achieved through the use of tangent-sphere co-ordinates, recently employed by O'Neill (1969) to obtain an exact solution for the special case of two equal-sized spheres in contact settling with equal velocities perpendicular to their lines of centres, and will include as a particular case the solution given recently by Wakiya (1971) for equal-sized spheres in a simple shear flow. The present solution will then be used to evaluate the particle stress caused by the presence of the aggregate and thereby the contribution of the latter to the rate of energy dissipation in a dilute suspension. It is also worth remarking that, on the basis of this solution, the net hydrodynamic force and torque on each sphere are both predicted to be, generally, non-zero, and therefore must be balanced by the force exerted by one sphere on the other at their point of contact.

Finally, it will be seen that, although the resulting equation for the angular velocity of solid-body rotation is similar to that of an equivalent spheroid with a constant axis ratio which is a function only of the size ratio of the spheres, the solution also leads to a finite drift velocity for the aggregate that depends on the size ratio of the spheres and their orientation in the flow field.

We proceed now to describe the formal solution of the Stokes equations for two spheres in contact.

2. The equations of motion

Under the assumptions of incompressible creeping flow and the existence of a quasi-steady solution, the disturbance caused by the presence of a touching-spheres aggregate satisfies the Stokes equations

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{\partial p}{\partial x_i}, \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where u_i and p are, respectively, the velocity and the pressure relative to the corresponding variables u_{0i} and p_0 of the undisturbed stream, and μ is the fluid viscosity. The appropriate boundary conditions are that

$$u_i, p \to 0 \quad \text{as} \quad (x_k x_k)^{\frac{1}{2}} \to \infty \tag{2.3}$$

together with the no-slip requirement on the surface of the spheres. The latter results in a rather simple expression, because, as would be expected on the basis of O'Neill & Majumdar's (1970) analysis regarding the motion of two spheres separated by a small gap, and as has already been established experimentally by Mason and co-workers, no relative rotation or translation between the touching spheres occurs under creeping-flow conditions. Consequently, the pair will be in rigid-body rotation with angular velocity Ω_i and the remaining boundary condition reduces to

$$u_i = U_i + \epsilon_{ijk}\Omega_j x_k - u_{0i} \quad \text{on the surface,} \tag{2.4}$$

where U_i is the velocity of the point of contact and p_i is a unit vector pointing along the direction of the line of centres from the smaller sphere towards the larger.

A solution to (2.1) and (2.2) subject to (2.3) and (2.4) can now be given generally for any set of curvilinear co-ordinates conjugate to the cylindrical system. Specifically, it can be shown from the work of Lin *et al.* (1970) that a formal solution to (2.1), with p satisfying Laplace's equation, is

$$\left.\begin{aligned} u_r &= \frac{1}{2}\sum_{m=0}^{\infty}\{(r\pi_m + u_m + v_m)\sin m\phi + (r\pi_{-m} + u_{-m} + v_{-m})\cos m\phi\},\\ u_\phi &= \frac{1}{2}\sum_{m=0}^{\infty}\left\{\frac{1}{m}(v_m - u_m)\frac{\partial}{\partial\phi}\sin m\phi + \frac{1}{m}(v_{-m} - u_{-m})\frac{\partial}{\partial\phi}\cos m\phi\right\},\\ u_z &= \frac{1}{2}\sum_{m=0}^{\infty}\{(z\pi_m + w_m)\sin m\phi + (z\pi_{-m} + w_{-m})\cos m\phi\},\\ p &= \mu\sum_{m=0}^{\infty}\{\pi_m \sin m\phi + \pi_{-m}\cos m\phi\}, \end{aligned}\right\} \tag{2.5}$$

where u_r, u_ϕ and u_z are the three cylindrical velocity components, and π_m, u_m, v_m and w_m are auxiliary harmonic functions of the other (than ϕ) two curvilinear co-ordinates.

The application of the general form of the solution to the case of touching spheres is facilitated through the use of the tangent-sphere co-ordinate system given by

$$z + ir = i(\eta + i\xi)^{-1} \tag{2.6}$$

and illustrated in figure 1. Here, the appropriate harmonic functions are

$$w_m = (\xi^2 + \eta^2)^{\frac{1}{2}}\int_0^\infty \{A_m(\nu)\cosh\nu\xi + B_m(\nu)\sinh\nu\xi\} J_{|m|}(\nu\eta)\, d\nu \tag{2.7}$$

together with similar expressions for π_m, v_m and u_m, in which the set $(A_m, B_m, J_{|m|})$ is replaced by, respectively, $(C_m, D_m, J_{|m|})$, $(E_m, F_m, J_{|m|-1})$ and $(G_m, H_m, J_{|m|+1})$. In (2.7), the $J_{|m|}(\nu\eta)$ are Bessel functions of the first kind and order $|m|$, and $A_m(\nu)$–$H_m(\nu)$ are continuous functions of the parameter ν. For further discussion it is understood that the integer m assumes absolute values except as a subscript to the auxiliary functions A–H. The latter are to be determined by requiring that (2.5)–(2.7) satisfy (2.2) plus the boundary conditions (2.3) and (2.4).

Substitution of (2.5) into (2.2) leads next to

$$\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\pi_m + \left(\frac{\partial}{\partial r} + \frac{m+1}{r}\right)u_m + \left(\frac{\partial}{\partial r} - \frac{m-1}{r}\right)v_m + 2\frac{\partial}{\partial z}w_m = 0, \tag{2.8}$$

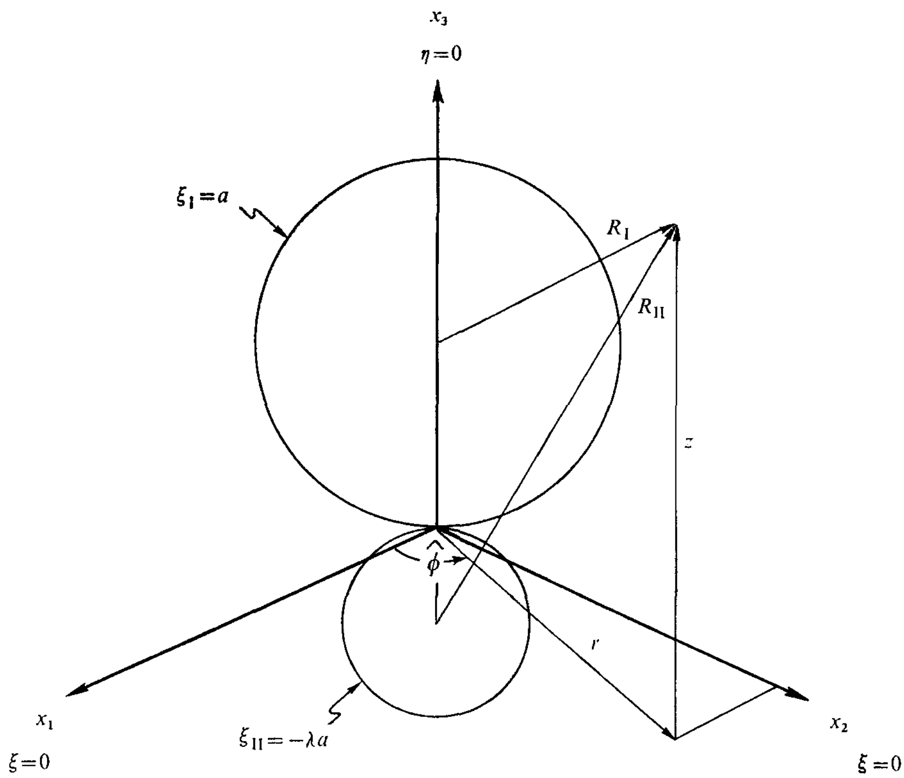

FIGURE 1. The various co-ordinate systems used. The radii of the spheres are $a_N = (2\xi_N)^{-1}$ respectively.

which, in view of (2.7) and the recurrence relations among the Bessel functions $J_m(\nu\eta)$, reduces to the two sets of differential equations involving the various auxiliary coefficients

$$\{\nu d+3\}\begin{pmatrix}C_m\\D_m\end{pmatrix}-2\left\{\nu d^2+d-\frac{m^2}{\nu}\right\}\begin{pmatrix}B_m\\A_m\end{pmatrix}+\left\{\nu d^2+2(m+1)\,d+\frac{m(m+1)}{\nu}\right\}\begin{pmatrix}G_m\\H_m\end{pmatrix}$$
$$-\left\{\nu d^2-2(m-1)+\frac{m(m-1)}{\nu}\right\}\begin{pmatrix}E_m\\F_m\end{pmatrix}=0, \quad (2.9)$$

where $d \equiv d/d\nu$. Similarly, it is possible to show that, on account of (2.4), the functions $C_m(\nu)$–$H_m(\nu)$ may be expressed solely in terms of $A_m(\nu)$ and $B_m(\nu)$ plus some known integrals arising from U_i and Ω_i and the undisturbed velocity u_{0i} evaluated at the surface. (Explicit expressions for arbitrary-sized spheres and a linear undisturbed flow field have been deposited with the editors of this journal and are available from them upon request.) As a result, in lieu of (2.9), two linear coupled inhomogeneous ordinary differential equations in A_m and B_m are obtained of the form

$$\mathscr{L}_m\{A_m, B_m\} = \mathscr{A}_m(\nu), \quad \mathscr{M}_m\{A_m, B_m\} = \mathscr{B}_m(\nu), \quad (2.10)$$

where $\mathscr{L}_m$ and $\mathscr{M}_m$ are linear operators, while $\mathscr{A}_m$ and $\mathscr{B}_m$ are known functions arising from the known integrals appearing in the expressions which relate

C_m–H_m to A_m and B_m. It should be noted that, for equal-sized spheres, the linking terms containing B_m in $\mathscr{L}_m$ and A_m in $\mathscr{M}_m$ vanish, hence, in this case, the two sets become uncoupled.

The relations between the various coefficients that result from using (2.4) also lead to the conditions

$$d^2A_m(\nu) = O(\nu^{-m}), \quad d^2B_m(\nu) = O(\nu^{-m-1}) \quad \text{as} \quad \nu \to 0, \tag{2.11}$$

which are more restrictive than those arising from (2.7). The above, plus the requirement that $A_m(\nu)$ and $B_m(\nu)$ decay exponentially for large ν, which follows naturally from (2.3) and (2.7), then become the boundary conditions for the numerical solution of (2.10).

We proceed next to evaluate the solution to the creeping-flow equations for the particular case of a freely suspended aggregate.

3. The freely suspended aggregate

The free suspension conditions manifest themselves through the requirement that the instantaneous net force and torque on the aggregate be zero. To calculate these quantities use is made of the expression for the stress force as given by Lamb (1945, p. 596):

$$\sigma_{ij}n_j = -pn_i + \mu\left(\frac{\partial}{\partial R_N} - \frac{1}{a_N}\right)u_i + \frac{\mu}{a_N}\frac{\partial}{\partial x_i}[(x_j \mp a_N p_j)u_j] \quad (N = \mathrm{I}, \mathrm{II}), \tag{3.1}$$

where R_N refers to the radial co-ordinate originating at the centre of the sphere whose radius is a_N. The total hydrodynamic force on each sphere is then

$$F_i^N = \int_{A_N} \sigma_{ij}n_j dA_N \quad (A_N = \text{surface of } N\text{th sphere}), \tag{3.2}$$

the Cartesian components of which for arbitrary-sized spheres, are, relative to the co-ordinate system of figure 1,

$$\left.\begin{aligned} F_1^N &= -2\pi\mu\int_0^\infty [E_{-1}(\nu) \pm F_{-1}(\nu)]\,d\nu, \\ F_2^N &= -2\pi\mu\int_0^\infty [E_{+1}(\nu) \pm F_{+1}(\nu)]\,d\nu, \\ F_3^N &= -4\pi\mu\int_0^\infty [A_0(\nu) \pm B_0(\nu)]\,d\nu. \end{aligned}\right\} \tag{3.3}$$

Similarly, the components of the torque on each sphere about the point of contact,

$$T_i^N = \epsilon_{ijk}\int_{A_N} \sigma_{jl}x_k n_l dA_N, \tag{3.4}$$

are

$$\left.\begin{aligned} T_1^N &= 2\pi\mu\int_0^\infty \nu[F_{+1}(\nu) \pm E_{+1}(\nu)]\,d\nu, \\ T_2^N &= -2\pi\mu\int_0^\infty \nu[F_{-1}(\nu) \pm E_{-1}(\nu)]\,d\nu, \\ T_3^N &= \pi\mu\int_0^\infty \nu[E_0(\nu) \pm F_0(\nu) + G_0(\nu) \pm H_0(\nu)]\,d\nu. \end{aligned}\right\} \tag{3.5}$$

In (3.1)–(3.5) the upper sign refers to sphere I and the lower to sphere II. (Explicit results for the forces and torques appearing in (3.3) and (3.5) are given in the appendix.)

Because of its relevance to rheological studies, a linear shear flow will be chosen next for the undisturbed flow field, with its origin set at the point of contact. Thus, with $2\omega_i$ and e_{ij}, respectively, the local vorticity and rate-of-strain tensor of the undisturbed flow,

$$u_{0i} = (e_{ik} + \epsilon_{ijk}\,\omega_j)\,x_k, \tag{3.6}$$

in which case $A_m(\nu)$ and $B_m(\nu)$ plus all the auxiliary functions $C_m(\nu)$–$H_m(\nu)$ can be shown to vanish identically for $m > 2$. Also, the requirement that the components of the total force and torque acting on the aggregate vanish, plus the fact that the orientation of the aggregate is specified uniquely by the unit vector p_i, leads to a pair of simple expressions relating $\Omega_i - \omega_i$, the angular velocity of solid-body rotation of the aggregate relative to that of the free stream, and U_i, the velocity of the point of contact, to e_{ij}, the rate-of-strain tensor of the undisturbed flow.

We note first that, since the axial vector $\Omega_i - \omega_i$ is linear in e_{ij} and is a function also of p_i, we have, following Bretherton (1962), that

$$\Omega_i - \omega_i = \tfrac{1}{2}b(\lambda)\,\epsilon_{ijk}(e_{km}p_m p_j - e_{jm}p_m p_k), \tag{3.7}$$

where $b(\lambda)$ is a scalar function of λ ($\equiv a_{\mathrm{I}}/a_{\mathrm{II}}$), the size ratio of the spheres. Clearly, for the particular orientation of the co-ordinate system shown in figure 1, $p_i = \delta_{i3}$ and, therefore,

$$\Omega_1 - \omega_1 = -\frac{r_e^2-1}{r_e^2+1}e_{23}, \quad \Omega_2 - \omega_2 = \frac{r_e^2-1}{r_e^2+1}e_{13}, \quad \Omega_3 - \omega_3 = 0, \tag{3.8}$$

where, for purposes of rendering (3.8) identical to Jeffery's (1922) well-known expression for the angular velocity of solid-body rotation of a spheroid, we have formally replaced $b(\lambda)$ by $(r_e^2-1)/(r_e^2+1)$, with r_e being referred to as the axis ratio of the equivalent spheroid. Similarly,

$$U_i = a_{\mathrm{I}}\{\gamma_1(\lambda)\,e_{ij}p_j + \gamma_2(\lambda)\,e_{jk}p_j p_k p_i\}, \tag{3.9}$$

where again γ_1 and γ_2 are functions only of λ. It can also be shown, from (2.4), (3.7) and (3.9) that the velocity of a material point C_R located along the axis at a distance $a_{\mathrm{I}}\zeta$ from the point of contact and with

$$\zeta(\lambda) = \gamma_1/(1-b) \tag{3.10}$$

consists of two terms: the undisturbed flow velocity at C_R

$$a_{\mathrm{I}}\zeta(e_{ij}p_j + \epsilon_{ijk}\,\omega_j p_k),$$

plus a 'drift' velocity

$$a_{\mathrm{I}}\beta e_{jk}p_j p_k p_i \quad (\beta(\lambda) \equiv \gamma_1 + \gamma_2), \tag{3.11}$$

the latter being along the axis. It then follows that if the origin of the undisturbed shear field were placed at C_R, henceforth referred to as the *centre of free rotation*,

λ	r_e (3.8)	ζ (3.10)	β (3.11)	h_1 (3.16)	h_2 (3.16)	β_1 (4.3)	β_2 (4.3)	β_3 (4.3)
1·00	1·982	0	0	4·463	7·767	5·255	4·764	0·9638
1·01	1·981	0·0118	0·0059	4·449	7·520	5·265	4·764	0·9580
1·20	1·940	0·2000	0·0950	3·844	5·629	4·943	4·773	0·9855
1·50	1·798	0·4088	0·1610	3·356	3·383	3·833	4·805	1·053
2·00	1·559	0·6259	0·1694	2·879	2·253	2·045	4·861	1·047
3·00	1·279	0·8258	0·1145	2·116	−0·2359	0·4174	4·929	0·7673
5·00	1·094	0·9424	0·0479	1·190	−0·5901	−0·1054	4·976	0·3539
10·0	1·017	0·9893	0·0103	0·3853	−0·2820	−0·0433	4·996	0·0728
∞	1·000	1·000	0	0	0	0	5·000	0

TABLE 1. A summary of the coefficients appearing throughout the paper as a function of the size ratio of the spheres. The numbers in parentheses refer to the equations in which the constants are defined.

the velocity of any material point would equal the drift velocity given by (3.11) plus a velocity of solid-body rotation

$$\epsilon_{ijk}\,\Omega_j(x_k - \zeta a_{\mathrm{I}} p_k),$$

where $x_k - \zeta a_{\mathrm{I}} p_k$ is the position vector originating at C_R.

The three scalars r_e, ζ and β, all of which depend only on the size ratio λ, can now be determined from the requirement of zero torque and force on the aggregate:

$$F_i^{\mathrm{I}} + F_i^{\mathrm{II}} = 0, \quad T_j^{\mathrm{I}} + T_j^{\mathrm{II}} = 0. \tag{3.12}$$

Because of (3.3), (3.5) and (3.8), however, and the fact that, in the co-ordinate system of figure 1, $U_1 = a_{\mathrm{I}}\zeta(1-b)\,e_{13}$ and $U_2 = a_{\mathrm{I}}\zeta(1-b)\,e_{23}$, it is only necessary, for purposes of obtaining r_e and ζ, to consider (3.12) with $i = 1, j = 2$ (or $i = 2$, $j = 1$), which is equivalent to requiring that

$$\int_0^\infty E_m(\nu)\,d\nu = \int_0^\infty \nu F_m(\nu)\,d\nu = 0 \quad (m = \pm 1).$$

It is then easy to see that the remaining relations in (3.12) are automatically satisfied, except for the condition $F_3^{\mathrm{I}} + F_3^{\mathrm{II}} = 0$, and the functions $r_e(\lambda)$ and $\zeta(\lambda)$ thus computed are given numerically in table 1.

Unfortunately, the present method of solving (2.1) and (2.2) applies only when $U_3 = 0$ since otherwise improper integrals are encountered in the expressions for C_0 and D_0 which diverge as $\eta \to \infty$. Thus, in order to satisfy the requirement that the component of the net force along the axial direction p_i should also vanish, and thereby determine $\beta(\lambda)$, it is necessary to superimpose on to the present solution (with $U_3 = 0$) the one given recently by Goren (1970) for the axisymmetric translation of two touching spheres in a quiescent fluid. The resulting numerical values for $\beta(\lambda)$ are generally found to be non-zero (see table 1), a fact which merely reflects the lack of fore-and-aft symmetry in the geometry of the aggregate when the spheres are of unequal size. It is important that the presence of this non-zero drift should not be overlooked, since it is clear that the free motion of the aggregate in a shear flow is not uniquely described by 'Jeffery's orbit equations' (3.7). Rather, the trajectories marked by the tip of p_i must be determined from (3.7) in conjunction with (3.6) and (3.9).

This can be illustrated, for example, by considering the special case in which the undisturbed flow is a simple shear of the form $u_2' = \gamma x_3'$, where u_i' and x_i' are, respectively, the velocity and position vectors relative to a fixed co-ordinate system and γ is the strength of the shear. Then, since

$$p_1 = \cos\theta, \quad p_2 = \sin\phi\sin\theta, \quad p_3 = \cos\phi\sin\theta,$$

where ϕ and θ are the familiar Euler angles, we have from (3.9) that

$$\left.\begin{aligned} dy_1/dt &= \beta(\lambda)\gamma\sin\phi\cos\phi\sin^2\theta\cos\theta, \\ dy_2/dt &= \gamma\{y_3+\beta(\lambda)\sin^2\phi\cos\phi\sin^3\theta\}, \\ dy_3/dt &= \beta(\lambda)\gamma\sin\phi\cos^2\phi\sin^3\theta, \end{aligned}\right\} \tag{3.13}$$

where $a_I y_i$ denotes the position of the centre of free rotation, together with Jeffery's (1922) expressions

$$\tan\phi = r_e\tan\frac{\gamma t}{r_e+(1/r_e)}, \quad \tan\theta = \frac{Cr_e}{(r_e^2\cos^2\phi+\sin^2\phi)^{\frac{1}{2}}}, \tag{3.14}$$

in which C is the so-called 'orbital constant'. This simple system has a general solution of the form

$$\left.\begin{aligned} y_1 &= k_1-\beta(\lambda)\frac{r_e^2+1}{r_e^2-1}\cos\theta, \\ y_2 &= k_2+k_3\frac{r_e^2+1}{r_e}\tan^{-1}\left(\frac{1}{r_e}\tan\phi\right) \\ &\quad +\beta(\lambda)\frac{(r_e^2+1)^2}{(r_e^2-1)^{\frac{3}{2}}}\left\{\int_0^\phi \tan^{-1}((r_e^2-1)^{\frac{1}{2}}\cos\phi\sin\theta)\frac{d\phi}{r_e^2\cos^2\phi+\sin^2\phi}\right. \\ &\quad \left. -(r_e^2-1)^{\frac{1}{2}}\sin\phi\sin\theta-\tanh^{-1}[r_e^{-1}(r_e^2-1)^{\frac{1}{2}}\sin\phi\sin\theta]\right\}, \\ y_3 &= k_3-\beta(\lambda)\frac{r_e^2+1}{r_e^2-1}\left\{\cos\phi\sin\theta-\frac{1}{(r_e^2-1)^{\frac{1}{2}}}\tan^{-1}[(r_e^2-1)^{\frac{1}{2}}\cos\phi\sin\theta]\right\}, \end{aligned}\right\} \tag{3.15}$$

where the k's are constants of integration. Since, obviously, both k_1 and k_2 may be set arbitrarily without affecting the shape of the trajectories $y_i(t)$, it is evident that the latter will depend only on C and on k_3 which equals $y_3(0)$ if ϕ is set initially equal to $\frac{1}{2}\pi$. It is also clear that the aggregate will experience a net translation in the positive-x_2' direction when $y_3(0) > 0$, and vice versa, and that it will move in closed orbits when $y_3(0) = 0$. Examples for the case of the two-dimensional motion of p_i in the x_2', x_3' plane ($C = \infty$, $\theta = \frac{1}{2}\pi$) are illustrated in figures 2 and 3.† Shown in figure 2 are the trajectories of the centre of free rotation of an aggregate with size ratio $\lambda = 2$ for various initial conditions, while figure 3 depicts typical closed orbits of the centre of free rotation for aggregates with low and high size ratios. The trajectories of any other material point may also be inferred from these graphs using, as additional information, the instantaneous

† A direct numerical integration of (3.13) was found to be faster than the evaluation of (3.15).

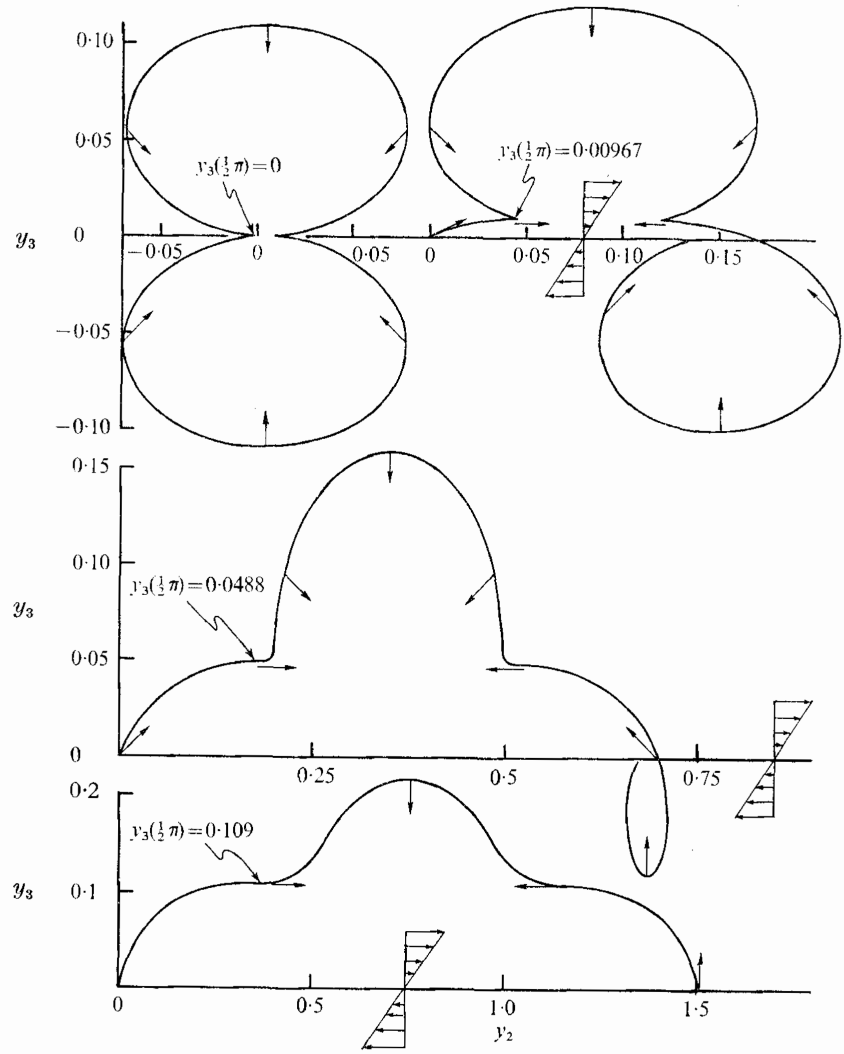

FIGURE 2. Periodic trajectories of the centre of free rotation in simple shear flow for $\lambda = 2$, where λ is the size ratio of the spheres. The arrows indicate the orientation of the unit vector p_i at the corresponding points, and the co-ordinates are defined in (3.13). $y_3(\frac{1}{2}\pi)$ refers to the value of y_3 at $\phi = \frac{1}{2}\pi$.

orientation of the aggregate indicated by the direction of the arrows. Clearly, the initial orientation of the aggregate plays an important role in determining the trajectories of its centre of free rotation, which, apart from their striking and perhaps unexpected shapes, would be of interest in future studies dealing with problems that involve the interaction between pairs of such two-sphere aggregates.

It is finally of interest to evaluate F_i^N ($N = \mathrm{I}$ or II), the net hydrodynamic force acting on one of the two spheres when the aggregate itself is freely suspended. This force, being linear in e_{ij} and odd in p_i, must be of the form

$$F_i^N = \pm\pi\mu a_{\mathrm{I}}^2\{h_1(\lambda)\, e_{ik}p_k + h_2(\lambda)\, e_{jk}p_jp_kp_i\}, \tag{3.16}$$

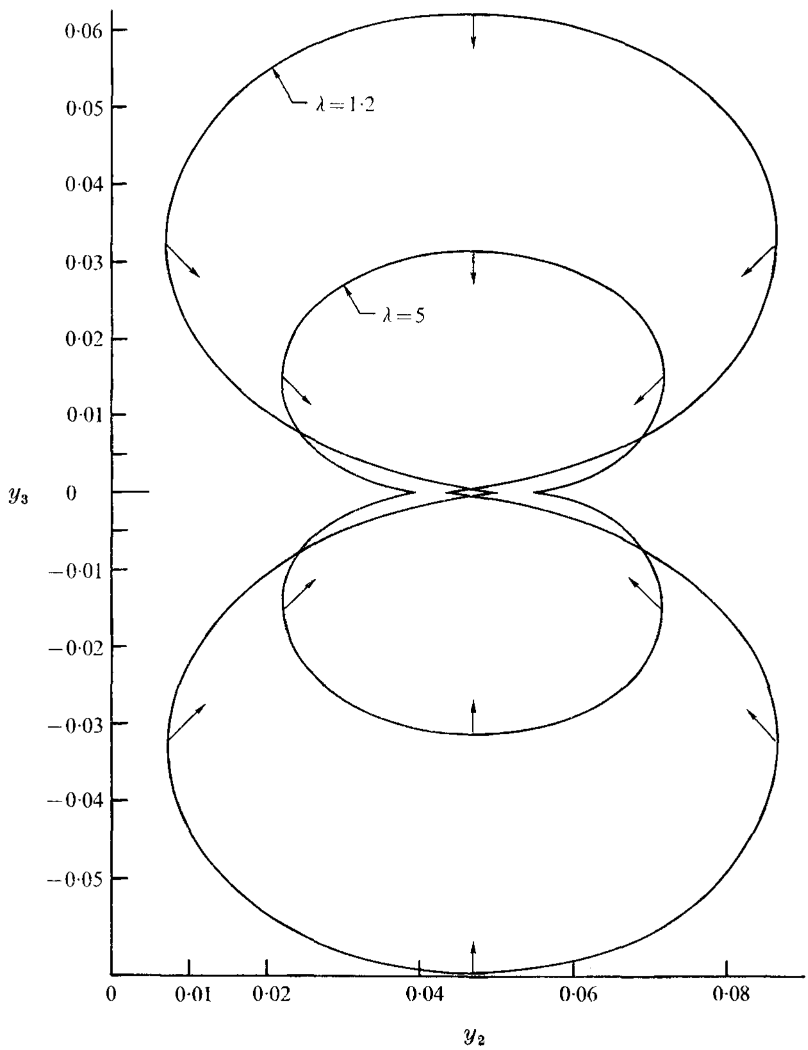

FIGURE 3. Closed orbits of the centre of rotation in simple shear flow for $\lambda = 1{\cdot}2$ and $\lambda = 5$. λ, y_3 and y_2 are as in figure 2.

where the plus sign has been chosen to refer to the larger of the two spheres ($N = \mathrm{I}$), of radius a_I, and in which the scalars h_1 and h_2, given in table 1, have been determined from (3.3).

4. Particle stress in dilute suspensions

One of the most important consequences of the present exact solution for the creeping motion of a touching pair of unequal-sized spheres is that it leads to an expression for the contribution of the aggregate to the particle stress in dilute suspensions. Of course, a complete analysis of this subject should also consider those effects which determine the steady-state distribution of orbital orientations, such as inertia forces (Harper & Chang 1968) or Brownian diffusion (Leal & Hinch 1971); however, these complications will not be included at this time.

It has been shown by Batchelor (1970) that, in the absence of inertia effects, the contribution of the particles in a suspension to the bulk stress of the equivalent continuum is given by

$$\Sigma_{ij}^{(p)} \equiv \Sigma\, \sigma_{ij}^{(p)} = \frac{1}{V}\Sigma \int_{V_0} \sigma_{ij}\, dV, \tag{4.1}$$

Σ denoting summation. The integration is over the volume of each particle and the summation is over all the particles within the representative averaging volume V, which must be large enough to contain a statistically significant number of particles yet small enough for the variation of the statistical properties over V to be negligible. Using the divergence theorem and in the absence of net forces or torques on the particles, $\sigma_{ij}^{(p)}$ can also be expressed as

$$\sigma_{ij}^{(p)} = \frac{1}{V}\int_{A_0} \sigma_{ik} x_j n_k\, dA, \tag{4.2}$$

where A_0 refers to the surface of each particle within V. For a dilute suspension, interaction among particles can be neglected, hence the contribution of a single aggregate to the particle stress can be obtained from (4.2) using the solution derived earlier.

Evidently, since $\sigma_{ij}^{(p)}$ is a function only of e_{ij} and of p_i, the unit vector specifying the orientation of the aggregate, it must satisfy Ericksen's (1959) constitutive relation for the stress of an anisotropic fluid whose anisotropy depends solely on a single 'director' p_i. Thus, for the special case of a stress which is linear in e_{ij} and isotropic when the latter is zero, we have that

$$\sigma_{ij}^{(p)} = \beta_0 \delta_{ij} + \frac{4\pi\mu a_{\mathrm{I}}^3}{3V}(1+\lambda^{-3})\{\beta_1 e_{km} p_k p_m p_i p_j + \beta_2 e_{ij} + \beta_3 (e_{ik} p_k p_j + e_{jk} p_k p_i)\} \tag{4.3}$$

and $$dp_i/dt = \epsilon_{ijk}\Omega_j p_k, \quad \text{with } \Omega_i \text{ given by (3.7)},$$

where β_1, β_2 and β_3 are functions only of λ. Furthermore, by applying our solution to (4.2), we obtain for the components of $\sigma_{ij}^{(p)}$ in the co-ordinate system of figure 1, in which $p_i = \delta_{i3}$,

$$\left.\begin{aligned} \sigma_{12}^{(p)} &= \frac{\pi\mu}{6V}\int_0^\infty \nu^2 C_2(\nu)\, d\nu = \frac{4\pi\mu a_{\mathrm{I}}^3}{3V}(1+\lambda^{-3})\beta_2 e_{12}, \\ \sigma_{13}^{(p)} &= -\frac{\pi\mu}{3V}\int_0^\infty \nu^2 D_{-1}(\nu)\, d\nu = \frac{4\pi\mu a_{\mathrm{I}}^3}{3V}(1+\lambda^{-3})(\beta_2+\beta_3) e_{13}, \\ \sigma_{22}^{(p)} - \tfrac{1}{3}\sigma_{kk}^{(p)} &= \frac{\pi\mu}{6V}\int_0^\infty \nu^2\{2C_0(\nu) - C_{-2}(\nu)\}\, d\nu \\ &= \frac{4\pi\mu a_{\mathrm{I}}^3}{3V}(1+\lambda^{-3})\{\beta_2 e_{22} - (e_{33}/3)(\beta_1 + 2\beta_3)\}, \end{aligned}\right\} \tag{4.4}$$

from which the β's shown in table 1 were determined.

Finally, as mentioned in the introduction, the present solution can also be used, in conjunction with some additional results pertaining to the statistical distribution and relative motion of a pair of spheres in a given flow field, for the purpose of deriving the $O(c^2)$ term in the constitutive equation for a non-dilute suspension of unequal-sized solid spheres. This extremely interesting and worthwhile analysis remains, however, to be carried out.

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Appendix

Although this paper is focused on the motion of a freely suspended aggregate, the present solution applies of course to particles which are not necessarily force free and couple free. Consider then the nine material tensors relating e_{ij}, $\Omega_i - \omega_i$ and $V_i - v_{i0}$ (the latter being the translational velocity of a material point P in the body relative to that of the undisturbed stream evaluated at P) to the total force F_i, the torque L_i and the stresslet S_{ij} (given, respectively, by (3.2), (3.4) and the integral in (4.2)), where L_i and S_{ij} are now expressed in terms of a position vector originating at P. The linearity of the creeping-flow equation requires that

$$\begin{pmatrix} F_i \\ L_i \\ S_{ij} \end{pmatrix} = \begin{pmatrix} A_{ik} & D'_{ik} & Q'_{ikl} \\ D''_{ik} & B_{ik} & R'_{ikl} \\ Q''_{ijk} & R''_{ijk} & C_{ijkl} \end{pmatrix} \begin{pmatrix} v_{k0} - V_k \\ \omega_k - \Omega_k \\ e_{kl} \end{pmatrix}, \qquad (\text{A } 1)$$

where, clearly, the values of $V_i - v_{i0}$, L_i and S_{ij} depend, in general, on the choice of P. Although, of course, the latter is arbitrary, the expressions for the nine tensors in (A 1) and, in particular, those for Q', Q'', R' and R'', simplify somewhat if P is chosen to coincide with the centre of free rotation C_R as defined following (3.10). With P then being at C_R and, in view of the axisymmetric shape of the aggregate, the fact that $\Omega_k - \omega_k$ must satisfy (3.7) under free suspension conditions, and the symmetries in the above grand matrix (discussed recently by Hinch 1972), it is easy to show that the various tensors in (A 1) depend only on ten constants and are of the form

$$\left.\begin{aligned} A_{ij} = a_1\delta_{ij} + a_2 p_i p_j, \quad B_{ij} = b_1 \delta_{ij} + b_2 p_i p_j, \\ D'_{ij} = -D''_{ij} = d_1 \epsilon_{ijk} p_k, \quad Q'_{ijk} = -q_1(\delta_{ik} p_j + \delta_{ij} p_k) - q_2 p_i p_j p_k, \\ Q''_{ijk} = -q_1(p_i \delta_{jk} + p_j \delta_{ik}) - q_2 p_1 p_j p_k, \quad R'_{ijk} = r_1(\epsilon_{ijl} p_l p_k + \epsilon_{ikl} p_l p_r), \\ R''_{ijk} = -r_1(\epsilon_{ikl} p_l p_j + \epsilon_{jkl} p_l p_i), \\ C_{ijkl} = c_1 p_i p_k p_j p_l + c_2 \delta_{ik} \delta_{jl} + c_3(p_i p_k \delta_{jl} + p_j p_k \delta_{il}), \end{aligned}\right\} \qquad (\text{A } 2)$$

where

$$\frac{q_1}{d_1} = \frac{r_1}{b_1} = \frac{1}{2}\frac{r_e^2 - 1}{r_e^2 + 1}. \qquad (\text{A } 3)$$

These coefficients are given in table 2 as functions of the size ratio λ. The corresponding coefficients in the expressions for the torques and forces on the individual sphere were also calculated and have been deposited with the editors of this journal, from whom they can be obtained upon request.

λ	$\frac{a_1}{\pi\mu a_I}$	$\frac{a_2}{\pi\mu a_I}$	$\frac{b_1}{\pi\mu a_I^3}$	$\frac{b_2}{\pi\mu a_I^3}$	$\frac{d_1}{\pi\mu a_I^2}$	$\frac{q_1}{\pi\mu a_I^2}$	$\frac{q_2}{\pi\mu a_I^2}$	$\frac{r_1}{\pi\mu a_I^3}$	$\frac{c_1}{V_0}$†	$\frac{c_2}{V_0}$†	$\frac{c_3}{V_0}$†
1·00	8·691	−0·951	29·92	−15·50	0	0	0	8·885	1·293	4·764	2·945
1·01	8·653	−0·949	29·48	−15·27	0·0220	0·0065	0·0323	8·748	1·309	4·764	2·936
1·20	7·996	−0·853	23·10	−11·65	0·3220	0·0934	0·4918	6·699	1·264	4·773	2·832
1·50	7·356	−0·705	17·52	−7·947	0·4758	0·1254	0·8195	4·619	1·013	4·805	2·463
2·00	6·797	−0·505	13·14	−4·590	0·4214	0·0878	0·8900	2·738	0·5232	4·861	1·808
3·00	6·353	−0·272	10·04	−1·925	0·2400	0·0289	0·6384	1·210	−0·0042	4·929	0·9780
5·00	6·113	−0·100	8·582	−0·568	0·0937	0·0042	0·2797	0·3840	−0·1510	4·976	0·3766
10·00	6·020	−0·019	8·093	−0·092	0·0206	0·00015	0·0611	0·0679	−0·0453	4·996	0·0736
∞	6·000	0	8·000	0	0	0	0	0	0	5·000	0

† $V_0 = \frac{4}{3}\pi a_I^3(1+\lambda^{-3})$, the volume of the pair.

TABLE 2. The constants appearing in the material tensors of (A 2).

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